

# A degree and forbidden subgraph conditon for a k-contractible edge

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### Abstract

An edge in a  $k$ -connected graph is said to be  $k$ -contractible if the contraction of it results in a  $k$ -connected graph. We say that  $k$ -connected graph  $G$  satisfies “degree-sum condition” if  $\sum_{x \in V(W)} \deg_G(x) \geq 3k + 2$  holds for any connected subgraph  $W$  of  $G$  with  $|W| = 3$ . Let  $k$  be an integer such that  $k \geq 5$ . We prove that if a  $k$ -connected graph with no  $K_1 + C_4$  satisfies degree-sum condition, then it has a  $k$ -contractible edge.

# 1 Introduction

In this paper, we deal with finite undirected graphs with neither self-loops nor multiple edges. For a graph  $G$ , let  $V(G)$  and  $E(G)$  denote the set of vertices of  $G$  and the set of edges of  $G$ , respectively. Let  $V_k(G)$  denote the set of vertices of degree  $k$  of  $G$ . We denote the degree of  $x \in V(G)$  by  $d_G(x)$ . Let  $\Delta(G)$  and  $\delta(G)$  denote the maximum degree of  $G$  and the minimum degree of  $G$ , respectively. For a vertex  $x \in V(G)$ , we denote by  $N_G(x)$  the neighborhood of  $x$  in  $G$ . Moreover, for a subset  $S \subseteq V(G)$ , let  $N_G(S) = \bigcup_{x \in S} N_G(x) - S$ . For nonintersecting vertex sets  $S, T \subseteq V(G)$ , we denote the set of edges between  $S$  and  $T$  by  $E_G(S, T)$ . We write  $E_G(x, S)$  for  $E_G(\{x\}, S)$ . When there is no ambiguity, we write  $V_k, N(x), N(S)$  and  $E(S, T)$  for  $V_k(G), N_G(x), N_G(S)$  and  $E_G(S, T)$ , respectively. For a vertex set  $S \subseteq V(G)$ , we let  $G[S]$  denote the subgraph induced by  $S$  in  $G$ . Let  $K_n, C_n$  and  $P_n$  denote the complete graph on  $n$  vertices, the cycle on  $n$  vertices and the path on  $n$  vertices, respectively. For a vertex set  $S \subseteq V(G)$ , we let  $G - S$  denote the graph obtained from  $G$  by deleting the vertices in  $S$  together with the edges incident with them; thus  $G - S = G[V(G) - S]$ . Let  $G$  be a connected graph. A subset  $S \subseteq V(G)$  is said to be a *cutset* of  $G$  if  $G - S$  is not connected. A cutset  $S$  is said to be a  $k$ -cutset if  $|S| = k$ . For a noncomplete connected graph  $G$ , the order of a minimum cutset of  $G$  is said to be the connectivity of  $G$  and the connectivity of  $G$  is denoted by  $\kappa(G)$ . Let  $k$  be an integer such that  $k \geq 2$  and let  $G$  be a graph with  $\kappa(G) = k$  and  $|V(G)| \geq k + 2$ . An edge  $e$  of  $G$  is said to be  $k$ -contractible if the contraction of the edge results in a  $k$ -connected graph. Note that, in the contraction, we replace each resulting pair of double edges by a simple edge. An edge which is not  $k$ -contractible is said to be  $k$ -noncontractible. Let  $xy \in E(G)$ . If  $N_G(x) \cap N_G(y) \cap V_k(G) \neq \emptyset$ , then  $xy$  is said to be trivially  $k$ -noncontractible. A  $k$ -connected graph is said to be *contraction-critically* if it has no  $k$ -contractible edge.

Every edge of a connected graph is 1-contractible. We observe that every 2-connected graph of order at least 4 has a 2-contractible edge. Tutte [8] showed that every 3-connected graph of order at least 5 has a 3-contractible edge. For  $k \geq 4$ , there are infinitely many contraction-critically  $k$ -connected graphs for each  $k$ . Hence, if  $k \geq 4$ , then we cannot expect the existence of a contractible edge in a  $k$ -connected graph with no condition. We let a  $k$ -sufficient condition stand for a condition for a  $k$ -connected graph to have a  $k$ -contractible edge.

There are some  $k$ -sufficient conditions involving degree. Edawa [3] proved the following minimum degree  $k$ -sufficient condition

**Theorem A.** *Let  $k \geq 2$  be an integer, and let  $G$  be a  $k$ -connected graph with  $\delta(G) \geq \lfloor \frac{5}{4}k \rfloor$ . Then  $G$  has a  $k$ -contractible edge, unless  $k \in \{2, 3\}$  and  $G$  is isomorphic to  $K_{k+1}$ .*

Krisell [5] extended Theorem A and proved the following degree sum  $k$ -sufficient condition.

**Theorem B.** *Let  $G$  be a  $k$ -connected graph for which  $d_G(v) + d_G(w) \geq 2\lfloor \frac{5}{4}k \rfloor - 1$  for any pair  $v, w$  of  $G$ . Then  $G$  contains a  $k$ -contractible edge.*

Solving a conjecture in [5], Su and Yuan [6] proved the following degree-sum  $k$ -sufficient condition which is an extension of Theorem B.

**Theorem C.** *Let  $G$  be a  $k$ -connected graph with  $k \geq 8$ . If  $d_G(v) + d_G(w) \geq 2\lfloor \frac{5}{4}k \rfloor - 1$  for any two adjacent vertices  $v, w$ , then  $G$  has a  $k$ -contractible edge.*

There are also some  $k$ -sufficient conditions involving forbidden subgraphs. We can see that "triangle-free" is a  $k$ -sufficient condition; Thomassen [7] pointed out this condition.

**Theorem D.** *Every  $k$ -connected graph with no triangle has  $k$ -contractible edge.*

A  $k$ -connected graph with no triangle has many  $k$ -contractible edges, which indicates the possible existence of a weaker condition involving forbidden subgraphs which guarantees a  $k$ -connected graph to have a  $k$ -contractible edge. In this direction, Kawarabayashi [4] showed the following, where  $K_4^-$  stands for the graph obtained from  $K_4$  by deleting one edge.

**Theorem E.** *For an odd integer  $k \geq 3$ , every  $k$ -connected graph with no  $K_4^-$  has a  $k$ -contractible edge.*

Since  $K_4^-$  contains a triangle, this is an extension of Theorem D when  $k$  is odd. We call the graph  $K_1 + 2K_2$  a *bowtie*. Ando, Kaneko, Kawarabayashi and Yoshimoto [1] proved that every  $k$ -connected graph with no bowtie has a  $k$ -contractible edge, which is also an extension of Theorem D.

**Theorem F.** *If a  $k$ -connected graph contains no  $K_1 + 2K_2$ , then it has a  $k$ -contractible edge.*

Theorems D, E and F deal with forbidden subgraph  $k$ -sufficient conditions. On the other hand, Theorem A gives a minimum degree  $k$ -sufficient condition and Theorem C gives a degree-sum  $k$ -sufficient condition. However, if we restrict ourselves to a class of graphs that satisfy some forbidden subgraph conditions, then we may relax the minimum degree bound in Theorem A and the degree-sum bound in Theorem C. The following forbidden subgraph condition relaxes the minimum degree bound (see Ando and Kawarabayashi [2]). Let  $K_5^-$  be the graph obtained from  $K_5$  by removing one edge.

**Theorem G.** *Let  $k$  be an integer such that  $k \geq 5$ . Let  $G$  be a  $k$ -connected graph which contains neither  $K_5^-$  nor  $5K_1 + P_3$ . If  $\delta(G) \geq k + 1$ , then  $G$  has a  $k$ -contractible edge.*

Note that if  $k \geq 5$ , then  $\lfloor \frac{5}{4}k \rfloor \geq k + 1$ . There is a  $k$ -regular contraction-critically  $k$ -connected graph which contains neither  $K_5^-$  nor  $5K_1 + P_3$ . Hence we cannot replace  $\delta(G) \geq k + 1$  by  $\delta(G) \geq k$  in Theorem G. In this sense, the minimum degree bound in Theorem G is sharp.

In the same direction, Yingqiu and Liang recently proved the following [9].

**Theorem H.** *For  $k \geq 5$ , let  $G$  be a  $k$ -connected graph which contains no  $K_1 + C_4$ . If  $d_G(v) + d_G(w) \geq 2k + 2$  for any two adjacent vertices  $v, w$ , then  $G$  has a  $k$ -contractible edge.*

In this paper we prove an extension of Theorem H which involves 3-degree-sum condition. For a connected subgraph  $W$  of  $G$ , we set  $d_G(W) = \sum_{x \in V(W)} d_G(x)$ .

**Theorem 1.** *Let  $k$  be an integer such that  $k \geq 5$ , and  $G$  be a  $k$ -connected graph which contains no  $K_1 + C_4$ . If  $d_G(W) \geq 3k + 2$  hold for any connected subgraph  $W$  of  $G$  with  $|W| = 3$ , then  $G$  has a  $k$ -contractible edge.*

To conclude this section, we give two examples of contraction critically  $k$ -connected graphs. The first one shows that, for each  $k \geq 5$ , there is a contraction critically  $k$ -connected graph with no  $K_1 + C_4$  and  $d_G(W) \geq 3k + 1$  hold for any connected subgraph  $W$  of  $G$  with  $|W| = 3$ . The second shows that, for each even integer  $k \geq 8$ , there is a contraction critically  $k$ -connected graph such that it contains  $K_1 + C_4$  and  $d_G(W) \geq 3k + 2$  hold for any connected subgraph  $W$  of  $G$  with  $|W| = 3$ . There example shows that neither the degree-sum condition nor the forbidden subgraph condition of Theorem 1 can be dropped.

**Example 1.**

Let  $G$  be the graph illustrated in Figure 1. Then we observe that each edge of  $G$  is trivially noncontractible and  $G$  contains no  $K_1 + C_4$ . Hence  $G$  is a contraction-critically 5-connected graph which satisfies the forbidden subgraph condition of Theorem 1.

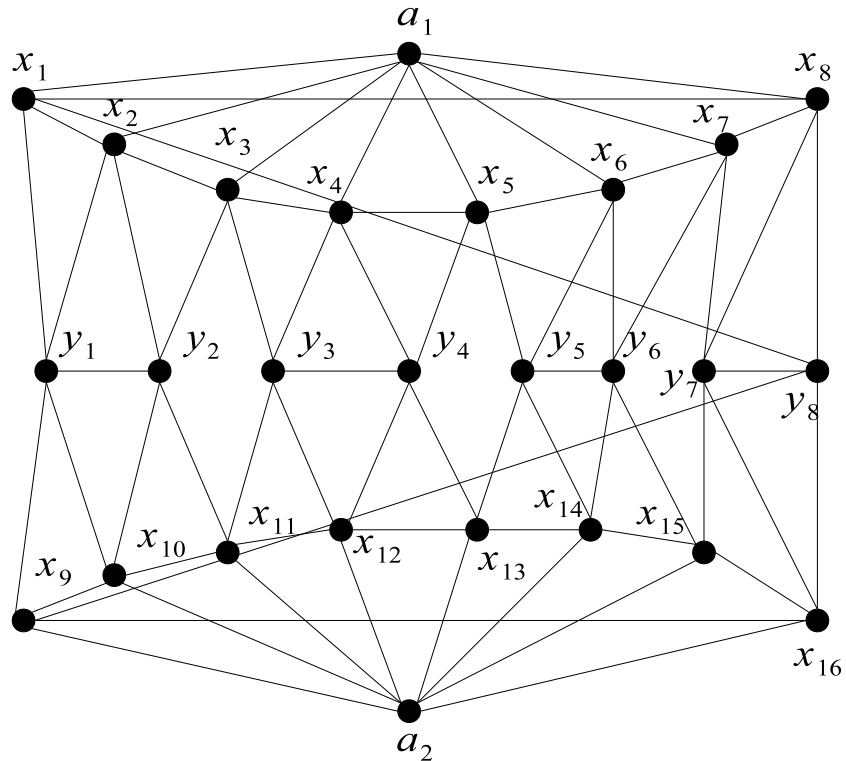


Figure 1: graph  $G$

**Example 2.**

Let  $G$  be the graph illustrated in Figure 2. Let  $S_1 = \{x_1, x_2, x_3, x_4\}$ ,  $S_2 = \{a_1, a_2, a_3, a_4\}$  and  $S_3 = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ . Then we observe  $S = S_1 \cup S_2$  is a 8-cutset of  $G$ . We also observe that each vertex of  $S_1$  has degree 8 and end vertex of  $V(G) - S_1$  has degree more than 10. Since each vertex of  $S_1$  has degree 8, we see that each edge of  $E_G(S_1, S_3) \cup E(G[S_3])$  is trivially noncontractible. Since  $E(G) = E_G(S_1, S_3) \cup E(G[S_3]) \cup E(G[S])$  and  $S$  is an 8-cutset of  $G$ , we see that  $G$  has no contractible edge. Let  $W$  be a connected subgraph of  $G$  such that  $|W| = 3$ . Since  $W$  is connected, we observe that  $|V(W) \cap S_1| \leq 2$  and  $|V(W) \cap (S_2 \cup S_3)| \geq 1$ , which implies  $d_G(W) \geq 3 \times 8 + 2 = 24$ . Hence  $G$  is a contraction-critically 8-connected graph which satisfies the degree-sum condition of Theorem 1.

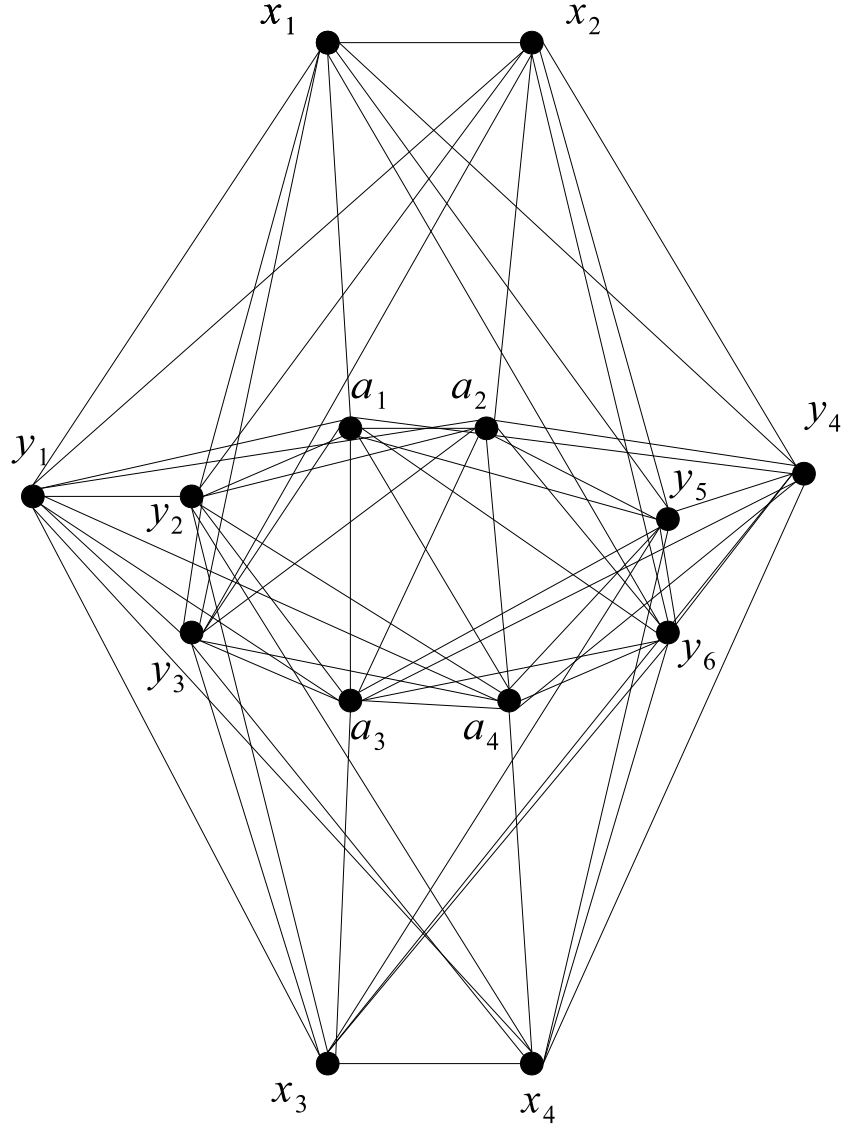


Figure 2: graph  $G$

Similarly we can construct a contraction-critically  $k$ -connected graph which satisfied the degree-sum condition of Theorem 1, for any  $k$  such that  $k \geq 9$

## 2 Prerequisites

In this section we give some more definitions and preliminary results.

For a graph  $G$ , we write  $|G|$  for  $|V(G)|$ . For a subgraph  $H$  of graph  $G$ , when there is no ambiguity, we sometimes write simply  $H$  for  $V(H)$ . So  $N_G(H)$  and  $G - H$  mean  $N_G(V(H))$  and  $G - V(H)$ , respectively. For a vertex  $x$  of  $G$ , we write  $E_G(x)$  for  $E_G(\{x\}, V(G) - \{x\})$ . When there is no ambiguity we write  $E(x)$  for  $E_G(x)$ . Hence  $E(x)$  stands for the set of edges incident with  $x$ . We denote the set of end vertices of an edge  $e$  of  $G$  by  $V(e)$ . For a connected subgraph  $W$  of  $G$ , we denote the degree sum of  $W$  by  $d_G(W)$ , that is  $d_G(W) = \sum_{x \in V(W)} d_G(x)$ .

Let  $k$  be an integer such that  $k \geq 2$  and let  $G$  be a graph with  $\kappa(G) = k$ . Recall that an edge of  $G$  is said to be  $k$ -noncontractible if it is not  $k$ -contractible. An edge is  $k$ -noncontractible if and only if there is a  $k$ -cutset  $S$  which contained  $V(e) \subseteq S$ . An induced subgraph  $A$  of  $G$  is called a *fragment* if  $|N_G(A)| = k$  and  $V(G) - (A \cup N_G(A)) \neq \emptyset$ . In other words, a fragment  $A$  is nonempty union of components of  $G - S$  where  $S$  is a  $k$ -cutset of  $G$  such that  $V(G) - (A \cup S) \neq \emptyset$ . By the definition, if  $A$  is a fragment of  $G$ , then  $G - (A \cup N_G(A))$  is also a fragment of  $G$ . Let  $\bar{A}$  stand for  $G - (A \cup N_G(A))$ . For a  $k$ -noncontractible edge  $e$  of  $G$ , a fragment  $A$  of  $G$  is said to be a fragment with respect to  $e$  if  $V(e) \subseteq N_G(A)$ . For a set of edges  $F \subseteq E(G)$ , we say that  $A$  is a fragment with respect to  $F$  if  $A$  is a fragment with respect to some  $e \in F$ . A fragment  $A$  with respect to  $F$  is said to be *minimum* if there is no fragment  $B$  other than  $A$  with respect to  $F$  such that  $|B| < |A|$ . If  $|A| = 1$  and  $|A| = 2$ , then a fragment  $A$  is said to be a *trivial fragment* and *2-fragment*, respectively. Moreover, if  $|A| \geq \lceil \frac{k+1}{2} \rceil$ , then a fragment  $A$  is said to be a large fragment.

For an edge  $e$  of  $G$ , we write the cardinality of a minimum fragment with respect to  $e$  by  $\eta(e)$ . Namely,  $\eta(e) = \min\{|A| \mid A \text{ is a fragment with respect to } e\}$ . For a vertex  $x$  of  $G$ , we set  $\eta(x) = \max\{\eta(e) \mid e \in E(x)\}$ . By the definition  $\eta(x) = 1$  if and only if each edge of  $E(x)$  is trivially noncontractible. We set  $E_L(G) = \{e \in E(G) \mid \eta(e) \geq \lceil \frac{k+1}{2} \rceil\}$ . Hence  $E_L(G)$  is the set of edges  $e$  for which each fragment with respect to  $e$  is large.

In this paper we will frequently mention a specific graph,  $K_1 + C_4$ . For convenience we write  $(x_1, x_2x_3x_4x_5)$ , for a graph  $H \cong K_1 + c_4$ , with  $V(H) = \{x_1, x_2, x_3, x_4, x_5\}$ ,  $d_H(x_1) = 4$  and  $H[\{x_2, x_3, x_4, x_5\}] \cong C_4$  with  $x_2x_3, x_3x_4, x_4x_5, x_5x_2 \in E(H)$ .

In the proof Theorem 1 we will use the following Lemmas. The proofs of three Lemmas are not difficult.

**Lemma 1.** *Let  $G$  be a  $k$ -connected graph, and let  $A$  and  $B$  be fragments of  $G$ . Let  $S = N_G(A)$  and  $T = N_G(B)$ . Then the following hold. If  $A \cap B \neq \emptyset$ , then  $|S \cap B| \geq |\bar{A} \cap T|$ .*

**Lemma 2.** *Let  $G$  be a graph and let  $W$  be a subset of  $V(G)$ . Then  $\sum_{x \in V(G) - W} |N_G(x) \cap W| = \sum_{y \in W} d_G(y) - 2|E(W)|$ .*

Ando and Kawarabayashi proved the following Lemma 3 in [2], which plays a fundamental roll in the proof of Theorem 1.

**Lemma 3.** *Let  $G$  be a  $k$ -connected graph and let  $A$  be a minimum fragment with respect to  $E_L(G)$ . Suppose that  $A$  has a vertex  $x$  such that  $E(x) \cap E_L(G) \neq \emptyset$ . Then each edge in  $E(x) \cap E_L(G)$  is  $k$ -contractible.*



### 3 The proof of Theorem 1

In this section we give a proof of Theorem 1. Let  $k$  be an integer, such that  $k \geq 5$ . Assume that  $G$  is a contraction critically  $k$ -connected graph with no  $K_1 + C_4$  such that  $d_G(x) + d_G(y) + d_G(z) \geq 3k + 2$  hold for any connected subgraph  $W$  of  $G$  with  $V(W) = \{x, y, z\}$ . Since  $G$  is contraction critically  $k$ -connected, there is a  $k$ -cutset  $S$  of  $G$  such that  $e \in E(G[S])$  for every edge  $e$  of  $E(G)$ .

**Claim 1.** *Let  $S$  be a  $k$ -cutset of  $G$ , and let  $A$  be a minimum fragment of  $G - S$ . If  $|A| \notin \{1, 2\}$ , then  $|A| \geq \lceil \frac{k+1}{2} \rceil$ .*

**Proof.**

Clearly, we have either  $|A| \in \{1, 2\}$  or  $|A| \geq 3$ . We prove that if  $|A| \geq 3$ , then  $|A| \geq \lceil \frac{k+1}{2} \rceil$ . Set  $S = N_G(A)$ , and let  $H = G[S \cup V(A)]$ . For any  $w \in V(A)$ , we have  $N_G(w) = N_H(w)$ . Thus  $d_G(w) = d_H(w)$ . By the minimality of  $A$ ,  $A$  is connected. Since  $|A| \geq 3$ ,  $A$  contains a connected subgraph  $W$  with  $|W| = 3$ . Denote  $X = V(W) = \{x_1, x_2, x_3\}$ ,  $Q = V(H) - X$ . Since  $W$  is connected, without loss of generality we may assume that  $x_1x_2, x_2x_3 \in E(W)$ . We show  $|N_G(x_1) \cap N_G(x_2) \cap N_G(x_3)| \leq 1$ . Assume  $|N_G(x_1) \cap N_G(x_2) \cap N_G(x_3)| \geq 2$ , say  $N_G(x_1) \cap N_G(x_2) \cap N_G(x_3) \supset \{u, v\}$ . Then we observe that  $G[\{u, x_1, x_2, x_3, v\}] \supset K_1 + C_4 = (x_2, x_1ux_3v)$ , which contradicts the forbidden subgraph condition. Therefore  $|N_G(x_1) \cap N_G(x_2) \cap N_G(x_3)| \leq 1$ . So, at most one vertex of  $A \cup S - X$  is adjacent to all vertices of  $X$ , and every other vertex of  $V(H) - X$  is adjacent to at most two vertices of  $X$ . By the degree sum condition, we have that  $d_G(x_1) + d_G(x_2) + d_G(x_3) \geq 3k + 2$ . If  $x_1x_3 \notin E(G)$  then by Lemma 2, we have

$$\begin{aligned} 3k + 2 &\leq \sum_{x \in X} d_G(x) = 2|E(W)| + |E(X, Q)| \\ &\leq 2 \times 2 + 3 + 2(|A| + |S| - 3 - 1) \\ &= 2|A| + 2k + 1 \end{aligned}$$

Therefore  $2|A| \geq k + 3$ . Since  $|A|$  is an integer, we have  $|A| \geq \lceil \frac{k+3}{2} \rceil > \lceil \frac{k+1}{2} \rceil$ . If  $x_1x_3 \in E(G)$  then by the same arguments, we have

$$\begin{aligned} 3k + 2 &\leq \sum_{x \in X} d_G(x) = 2|E(W)| + |E(X, Q)| \\ &\leq 2 \times 3 + 3 + 2(|A| + |S| - 3 - 1) \\ &= 2|A| + 2k + 1 \end{aligned}$$

Therefore  $2|A| \geq k + 1$ . Since  $|A|$  is an integer, we have  $|A| \geq \lceil \frac{k+1}{2} \rceil$ . The proof of Claim 1 is completed.  $\square$

**Claim 2.** *Let  $x$  be a vertex of  $G$ . Let  $A$  be a connected 2-fragment with respect to  $E(x)$ . If  $E(x) \cap E_L(G) = \emptyset$ , then each fragment with respect to  $E_G(x, A)$  is trivial.*

**Proof.**

Let  $A = \{a, b\}$  and let  $S = N_G(A)$ . Let  $B$  be a fragment with respect to  $E_G(x, A)$  and let  $T = N_G(B)$ . Since  $E(x) \cap E_L(G) = \emptyset$ , Claim 1 assure us that  $|B| \in \{1, 2\}$ . Assume that  $|B| = 2$ , say  $B = \{u, v\}$ . We show  $b \in A \cap T$ . Assume that  $b \in A \cap B$ . Then  $A \cap \overline{B} = \emptyset$  since  $A = \{a, b\}$ . If  $S \cap \overline{B} = \emptyset$ , then  $|S \cap \overline{B}| < |A \cap T|$ . Hence Lemma 1 assure us that  $\overline{A} \cap \overline{B} = \emptyset$ , which implies that  $\overline{B} = \emptyset$ . This contradicts that the choice of  $B$ . Hence  $S \cap \overline{B} \neq \emptyset$ , and we observe  $d_G(b) = k$ , which implies  $A \cap B$  is a fragment with respect to  $E_G(x, A)$ . This contradicts the fact that  $B$  is minimum fragment. Hence  $b \notin A \cap B$ .

By the similar argument we can show that  $b \notin A \cap \overline{B}$ . Now it is shown that  $b \in A \cap T$ . We show  $|S \cap B| = 2$ , assume  $|S \cap B| \leq 1$ . Then  $|S \cap B| < |A \cap T|$  and Lemma 1 assure us that  $\overline{A} \cap B = \emptyset$ , which implies that  $B = S \cap B$  and  $|B| = |S \cap B| \leq 1$ . This contradicts the fact that  $|B| = 2$ . Hence we have  $|S \cap B| = 2$  and  $S \cap B = \{u, v\}$ . Since  $A$  is a connected fragment, we see that  $ab \in E(G)$ . Since  $B$  is a minimum fragment with respect to  $E_G(x, A)$ , we observe that  $uv \in E(G)$ . Since  $uv, ab \in E(G)$ , we see that  $G[\{a, b, u, v\}]$  has a 4-cycle, if  $|\{a, b, u, v\} \cap V_k(G)| \geq 2$ , we can find a connected subgraph  $W$  of  $G[\{a, b, u, v\}]$  such that  $|W| = 3$  and  $d_G(W) \leq 3k + 1$ , which contradicts the degree-sum condition. Hence  $|\{a, b, u, v\} \cap V_k(G)| \leq 1$ . Without loss of generality, we may assume that  $\{a, b, u\} \subseteq V_{k+1}(G)$ . Then we see that  $G[\{z, a, b, u, v\}] \supset K_1 + C_4 = (a, xbv u)$ , which contradicts the forbidden subgraph condition. Now Claim 2 is proved.  $\square$

**Claim 3.** *If  $x \in V_k(G) \cup V_{k+1}(G)$ . Then  $E(x) \cap E_L(G) \neq \emptyset$ .*

**Proof.**

At first, we show  $N_G(x) \cap V_k(G) \neq \emptyset$ . Assume that  $N_G(x) \cap V_k(G) = \emptyset$ . Let  $A$  be a minimum fragment with respect to  $E(x)$ . Since  $N_G(x) \cap V_k(G) = \emptyset$ , we observe that  $|A| \geq 2$ . Since  $E(x) \cap E_L(G) = \emptyset$ , Claim 1 assure us  $|A| \leq 2$ . Hence we see  $|A| = 2$ . Let  $B$  be a minimum fragment with respect to  $E_G(x, A)$ . Then by Claim 2 we have  $|B| = 1$ , which contradicts the assumption that  $N_G(x) \cap V_k(G) = \emptyset$ . Now it is showed that  $N_G(x) \cap V_k(G) \neq \emptyset$ .

Let  $y \in N_G(x) \cap V_k(G)$ . Let  $A$  be a minimum fragment with respect to  $xy$ . We show  $|A| \geq 2$ . Assume  $|A| = 1$ , say  $A = \{a\}$ . Then since  $x \in V_k(G) \cup V_{k+1}(G)$  and  $y, a \in V_k(G)$ , we have  $d_G(x) + d_G(y) + d_G(a) \leq 3k + 1$ , which contradicts the degree-sum condition. Now it is show that  $|A| \geq 2$ . Hence  $|A| = 2$ , say  $A = \{a, b\}$ . If  $\{a, b\} \cap V_k(G) \neq \emptyset$ , we see that  $d_G(y) + d_G(a) + d_G(b) \leq 3k + 1$ , which contradicts the degree-sum condition. Hence we observe that  $\{a, b\} \subset V_{k+1}(G)$ . Let  $B$  be a minimum fragment with respect to  $ya$ . Then Claim 2 assure us that  $|B| = 1$ , say  $B = \{u\}$ . Since  $u \in V_k(G)$  and  $x \in V_{k+1}$ , We observe that  $u \neq x$ . That  $d(G[\{y, u, a\}]) \leq 3k + 1$ , which contradicts the degree-sum condition. The proof of Claim 3 is completed.  $\square$

Now we proceed the proof of Theorem 1. Let  $A$  be a minimum fragment with respect to  $E_L(G)$  and let  $S = N_G(A)$ . Since  $G$  is contraction-critically, Lemma 3 assure us  $E(x) \cap E_L(G) = \emptyset$  for any  $x \in A$ . Hence by Claim 3 we observe that  $A \cap (V_k(G) \cup V_{k+1}(G)) = \emptyset$ .

**Claim 4.** *Let  $x \in A$  and let  $y \in N_G(x) \cap S$ . Let  $B$  be a minimum fragment with respect to  $xy$ . Then  $|B| = 1$ .*

**Proof.**

Let  $T = N_G(B)$ . Since  $x \in A$  and  $G$  is contraction critically, Lemma 3 assure us that  $E(x) \cap E_L(G) = \emptyset$ . Hence Claim 1 assure us  $|B| \in \{1, 2\}$ . Assume  $|B| = 2$ , say  $B = \{u, v\}$ . Since  $u, v \in V_k(G) \cup V_{k+1}(G)$ , Claim 3 assure us that  $A \cap B = \emptyset$ . If  $S \cap B = \emptyset$ , then  $|S \cap B| < |A \cap T|$ , which implies  $\bar{A} \cap B = \emptyset$  and  $B = \emptyset$ . This contradicts the choice of  $B$ . Hence  $S \cap B \neq \emptyset$ . We show  $\bar{A} \cap B = \emptyset$ , assume  $\bar{A} \cap B \neq \emptyset$ , say  $v \in \bar{A} \cap B$  and  $u \in S \cap B$ . Since  $|S \cap B| = |A \cap T|$ , we observe that  $|(S \cap B) \cup (S \cap T) \cup (\bar{A} \cap T)| = k$ , which implies  $v \in V_k(G)$ . Let  $C$  be a minimum fragment with respect to  $E_G(x, B)$ . Then, by Claim 2, we know that  $|C| = 1$ , say  $C = \{w\}$ . Since  $v \in \bar{A}$  and  $x \in A$ , we see that  $xv \notin E(G)$ , which implies  $w \neq v$ , since  $xw \in E(G)$ . Let  $W = G[\{u, v, w\}]$ . Since  $B$  is minimum, we know that  $uv \in E(G)$ , which implies that  $W$  is connected. Now we observe that  $d_G(W) = d_G(w) + d_G(u) + d_G(v) \leq 3k + 1$ , which contradicts the degree-sum condition. It is shown that  $\bar{A} \cap B = \emptyset$ . and  $B = S \cap B = \{u, v\}$

Since  $N_G(x) \cap B \neq \emptyset$ , say  $u = N_G(x)$ . Let  $C$  is a minimum fragment with respect to  $xu$ . Then by Claim 2, we have  $|C| = 1$ , say  $C = \{w\}$ . We show that  $w \neq v$ . Assume  $w = v$ . Then  $xv \in E(G)$  and  $v \in V_k(G)$ . If  $vy \in E(G)$ , then  $v \in N_G(x) \cap N_G(y) \cap V_k(G)$ , which contradicts that  $\eta(xy) = 1$ . Hence  $yv \notin E(G)$ , which implies  $yu \in E(G)$ . Since  $N_G(y) \cap B \neq \emptyset$ . Let  $C'$  be a minimum fragment with respect to  $xv$ . By Claim 2, we know that  $|C'| = 1$ , say  $C' = \{w'\}$ . If  $w' = u$ , then since  $yu \in E(G)$ , we see that  $u \in N_G(x) \cap N_G(y) \cap V_k(G)$ , which contradicts the assumption that  $\eta(xy) = 2$ . Hence  $w' \neq u$ . Then  $d_G(u) + d_G(v) + d_G(w') \leq 3k + 1$ , which contradicts the degree-sum condition. This contradiction proved that  $w \neq v$ .

If  $\{u, v\} \cap V_k(G) \neq \emptyset$ , then  $W = G[\{u, v, w\}]$  is a connected subgraph of  $G$  such that  $d_G(u) + d_G(v) + d_G(w) \leq 3k + 1$ , which contradicts the degree-sum condition. Hence  $\{u, v\} \subset V_{k+1}(G)$ .

We show  $|A \cap T| \geq 2$ . Assume  $|A \cap T| = 1$ . Then since  $|S \cap B| > |A \cap T| = 1$ , Lemma 1 assure us  $A \cap \bar{B} = \emptyset$ , which contradicts the fact that  $A$  is a large fragment with  $|A| \geq \lceil \frac{k+1}{2} \rceil \geq 3$ . It is shown that  $|A \cap T| \geq 2$ , say  $x' \in A \cap T - \{x\}$ . Let  $C'$  be a minimum fragment with respect to  $E_G(x', B)$ . Then, Claim 2 assures us that  $|C'| = 1$ , say  $C' = \{w'\}$ . Note that  $w, w' \in N_G(u)$ . Since  $u \in V_{k+1}(G)$ , if  $w \neq w'$ , then  $W = G[\{w, w', u\}]$  is a connected subgraph of  $G$  such that  $d_G(u) + d_G(w) + d_G(w') \leq 3k + 1$ , which contradicts the degree-sum condition. Hence  $w = w'$ . Then we observe that  $xw, x'w \in E(G)$  and  $\{x, w, x'\} \subset N_G(u) \cap N_G(v)$ , and we find a  $K_1 + C_4 = (w, xux'v)$  in  $G$  which contradicts the forbidden subgraph condition. The proof of Claim 4 is completed.  $\square$

**Claim 5.** *Let  $xy \in E(A)$ . Let  $B$  be a minimum fragment with respect to  $xy$ . Then  $|B| = 1$ .*

**Proof.**

Since  $x \in A$  and  $G$  is contraction critical, Lemma 3 assures us that  $E(x) \cap E_L(G) = \emptyset$ . Hence Claim 1 assure us  $|B| \in \{1, 2\}$ . Assume  $|B| = 2$ , say  $B = \{u, v\}$ . We show that  $u, v \in S \cap B$ . Since  $u, v \in V_k(G) \cup V_{k+1}(G)$ , Claim 3 assure us that  $A \cap B = \emptyset$ . If  $\bar{A} \cap B \neq \emptyset$ , then by Lemma 1, we have  $|S \cap B| < |A \cap T|$ , which implies  $\bar{A} \cap B = \emptyset$  and  $B = \emptyset$ . This contradicts the choice of  $B$ . Hence  $\bar{A} \cap B = \emptyset$ . It is shown that  $u, v \in S \cap B$ . We show  $\{u, v\} \subseteq V_{k+1}(G)$ . Assume,  $\{u, v\} \cap V_k(G) \neq \emptyset$ . Suppose  $u \in V_k(G)$ . Since  $xy, uv \in E(G)$  and note of  $N_G(x) \cap B$ ,  $N_G(y) \cap B$ ,  $N_G(u) \cap \{x, y\}$  and  $N_G(v) \cap \{x, y\}$  is empty, we observe that  $G[\{x, y\} \cup B]$  has a  $C_4$ . Without loss of generality we may assume

$C_4 = xuvy$ . Let  $C$  be a minimum fragment with respect to  $E_G(x, B)$ . Then Claim 2 assures us that  $|C| = 1$ , say  $C = \{w\}$ . If  $v = w$ , then  $v \in N_G(x) \cap N_G(y) \cap V_k(G)$ , which contradicts the assumption that  $\eta(xy) = 2$ . Hence  $v \neq w$ . Then  $W = G[\{u, v, w\}]$  is a connected subgraph of  $G$  such that  $d_G(u) + d_G(v) + d_G(w) \leq 3k + 1$ , which contradicts degree-sum condition. Now it is shown that  $\{u, v\} \subseteq V_{k+1}(G)$ .

Let  $D$  be a minimum fragment with respect to  $xu$ . Since Claim 4,  $|D| = 1$ , say  $D = \{w'\}$ . Since  $u, v \in V_{k+1}(G)$  and  $w' \in V_k(G)$ ,  $w' \neq u, v$ . Then we observe that  $\{x, y, w'\} \subset N_G(u) \cap N_G(v)$ , and we find a  $K_1 + C_4 = (x, yuw'v)$ , which contradicts the forbidden subgraph condition. This contradiction proves Claim 5  $\square$

Now we denote  $\mathcal{E} = \{f \in E(G) | V(f) \subset S \cap V_k(G)\}$ . Let  $f_1, f_2 \in \mathcal{E}$ . Since the degree-sum condition, if  $V(f_1) \cap V(f_2) \neq \emptyset$ , then  $f_1 = f_2$ .

**Claim 6.** *For each edge  $e \in E[A]$ , there exists an edge  $f \in \mathcal{E}$  such that  $G[V(e) \cup V(f)] \cong K_4$ .*

**Proof.**

Let  $B$  be a minimum fragment with respect to  $xy$ . By Claim 5,  $|B| = 1$ , say  $B = \{u\}$ , then  $u \in V_k(G) \cap S$  and  $xu, yu \in E(G)$ . Let  $C$  be a minimum fragment with respect to  $xu$ . By Claim 4,  $|C| = 1$ , say  $C = \{v\}$ . which implies  $v \in V_k(G) \cap S$  and  $xv, yv \in E(G)$ . Let  $D$  be a minimum fragment with respect to  $uy$ . By Claim 4, we have  $|D| = 1$ , say  $D = \{w\}$ , then  $w \in V_k(G) \cap S$  and  $xw, yw \in E(G)$ . If  $w \neq v$  then  $d(G[\{u, v, w\}]) = 3k$ , which contradicts the degree-sum condition. Hence  $w = v$ , which implies  $uv \in \mathcal{E}$ . We observe that  $G[\{x, y, u, v\}] \cong K_4$ . The proof of Claim 6 is completed.  $\square$

Claim 6 assures us that for each edge  $e \in E(A)$ , there is an edge  $f \in \mathcal{E}$  such that  $G[V(e) \cup V(f)] \cong K_4$ . For each  $e \in E(A)$  choosing a such edge  $f$  and setting  $\varphi(e) = f$ , we define a mapping  $\varphi$  from  $E(A)$  to  $\mathcal{E}$ .

We denote  $\text{dist}_A(e_1, e_2)$  be the distance between  $e_1$  and  $e_2$  in  $A$ . If  $\text{dist}_A(e_1, e_2) = 1$  then  $e_1$  and  $e_2$  have a common end vertex. If  $\text{dist}_A(e_1, e_2) = 2$ , then there is an edge  $e_3$  between a vertex in  $V(e_1)$  and a vertex in  $V(e_2)$ .

**Claim 7.** *Let  $e_1, e_2 \in E(A)$ . If  $e_1 \neq e_2$  and  $\text{dist}_A(e_1, e_2) \leq 2$ . then  $\varphi(e_1) \neq \varphi(e_2)$ .*

**Proof.**

Assume that  $\varphi(e_1) = \varphi(e_2) = f$ . Let  $V(e_1) = \{x_1, x_2\}$ ,  $V(e_2) = \{x_3, x_4\}$ , and  $V(f) = \{u, v\}$ . At first we consider the case that  $\text{dist}_A(e_1, e_2) = 1$ . Then  $V(e_1) \cap V(e_2) \neq \emptyset$ , without loss of generality we may assume that  $x_2 = x_3$ . Then we observe that  $x_1u, x_1v, x_2u, x_4u, x_4v \in E(G)$  and  $G[\{x_1, x_2, x_4, u, v\}] \supset K_1 + C_4 = (u, x_1vx_4x_2)$ , which contradicts the forbidden subgraph condition. Next we consider the case that  $\text{dist}_A(e_1, e_2) = 2$ . In this case all for vertices  $x_1, x_2, x_3, x_4$  any distinct and we may assume that  $x_2, x_3 \in E(G)$ . Then we observe that  $x_1u, x_1v, x_2u, x_3u, x_3v \in E(G)$  and  $G[\{x_1, x_2, x_3, u, v\}] \supset K_1 + C_4 = (u, x_1vx_3x_2)$  which contradicts the forbidden subgraph condition. Now Claim 7 is proved.  $\square$

**Claim 8.** *There is an edge  $xy \in E(A)$  such that  $d_A(x) + d_A(y) \geq \frac{2}{3}k + 3$ .*

**Proof.**

Let  $W$  be a subgraph of  $A$  which is isomorphic to a path of length 2. Let  $W = x_1x_2x_3$ .

**Subclaim 8.1.**  $d_A(W) = k + 5$

**Proof.**

Assume  $|N_G(x_1) \cap N_G(x_2) \cap N_G(x_3)| \geq 2$ , say  $u, v \in N_G(x_1) \cap N_G(x_2) \cap N_G(x_3)$ . Then we observe that there is a  $K_1 + C_4 = (x_2, x_1ux_3v)$ , which contradicts the forbidden subgraph condition. Hence  $|N_G(x_1) \cap N_G(x_2) \cap N_G(x_3)| \leq 1$ . Hence we have

$$\begin{aligned} |E_G(V(W), S)| &\leq 3 + 2(|S| - 1) \\ &= 2k + 1 \end{aligned}$$

Since  $\delta(A) \geq 2$ , we have  $d_G(W) \geq 3k + 6$ . Hence

$$\begin{aligned} d_A(W) &= d_G(W) - |E_G(V(W), S)| \\ &\geq (3k + 6) - (2k + 1) \\ &= k + 5 \end{aligned}$$

Now Subclaim 8.1 is proved.  $\square$

**Subclaim 8.2.**  $A$  contains a subgraph which is isomorphic to  $P_4$ .

**Proof.**

Assume  $A$  has no  $P_4$ . Then we see that  $\text{dist}(e, e') \leq 2$  for any two edge  $e, e' \in E(A)$ , and  $A$  has  $K_1, |A| - 1$ . Since  $A$  has  $K_1, |A| - 1$ , we see that  $\Delta(A) = |A| - 1$ . Since  $\delta(A) \geq 2$  and  $\Delta(A) = |A| - 1$ , we have  $2|E(A)| \leq (|A| - 1) + 2(|A| - 1) = 3|A| - 3$ . Since  $\text{dist}(e, e') \leq 2$  for any  $e, e' \in E(A)$ , Claim 7 assures us that  $|\varphi(E(A))| = |E(A)|$ . Hence  $|\mathcal{E}| \geq |\varphi(E(A))| = |E(A)|$ , which implies  $|S| \geq 2|\mathcal{E}| = 2|E(A)| \geq 3|A| - 3$ . Since  $|A| \geq \lceil \frac{k+1}{2} \rceil$ , we have

$$\begin{aligned} |S| &\geq 3\lceil \frac{k+1}{2} \rceil - 3 \\ &\geq k + \frac{k+3}{2} - 3 \\ &\geq k + 1 \end{aligned}$$

which contradicts the fact  $k = |S|$ . Now Subclaim 8.2 is proved.  $\square$

Subclaim 8.2 assure us that  $A$  contains  $P_4 = x_1x_2x_3x_4$ . Assume  $d_A(x_2) + d_A(x_3) \geq \frac{2}{3}k + 3$ , and  $d_A(x_3) + d_A(x_4) < \frac{2}{3}k + 3$ . Since  $d_A(x_2) + d_A(x_3) + d_A(x_4) \geq k + 5$ , we have  $d_A(x_4) \geq \frac{k}{3} + 2$ . Then  $d_A(x_3) < (\frac{2}{3}k + 3) - d_A(x_4) = \frac{k}{3} + 1$ . Hence  $d_A(x_1) + d_A(x_2) \geq (k + 5) - d_A(x_3) \geq \frac{2}{3}k + 3$ . Now Claim 8 is proved.  $\square$

Let  $A$  be a minimum fragment with respect to  $E_L(G)$ . Let  $N_G(A) = S$  and let  $x, y \in A$ . Since Claim 8, We observe that,  $d_A(x) + d_A(y) \geq \frac{2}{3}k + 3$ . By Claim 7, we have

$$\begin{aligned}
|S| \geq \left| \bigcup_{i=1}^{\frac{2}{3}k+2} V(\varphi(i)) \right| &= \sum_{i=1}^{\frac{2}{3}k+2} |V(\varphi(e_i))| \\
&= 2\left(\frac{2}{3}k + 2\right) \\
&> k
\end{aligned}$$

This contradicts that  $|S| = k$ . This is the final contradiction and proof of Theorem 1 is completed.

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